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Uniform Strong Unicity Constants for Subsets of $C(X)$

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1. INTRODUCTION

Let $C(X)$ denote the set of continuous real-valued functions on the compact metric space X with metric ρ. Let M be a Haar subspace of dimension n in $C(X)$ and, for a given $f \in C(X)$, let $B(f)$ or $B_M(f)$ denote the best uniform approximation to f from M. D. J. Newman and H. S. Shapiro [9] showed the existence of a constant $c > 0$ such that

$$\|f - m\| \geqslant \|f - B(f)\| + c\,\|B(f) - m\| \qquad \text{for all } m \in M.$$

The strong unicity constant $\gamma(f)$ is the largest such constant c and $0 < \gamma(f) \leqslant 1$.

This paper studies the existence of uniform strong unicity constants $\gamma > 0$ for subsets of $C(X)$. Cline [3] showed that there is no uniform strong unicity constant for all of $C(X)$ for X infinite and $n > 1$. Bartelt [1] showed that if X is finite then there is a uniform strong unicity constant for all of $C(X)$, and therefore we assume henceforth that X is infinite. It is known [5, 8, 10] that if the compact set $S \subseteq C[a, b]$ satisfies $S \cap M = \varnothing$, then S has a uniform strong unicity constant. As observed in [4], $C(X)$ has a uniform strong unicity constant when dim $M = 1$.

Results on uniform unicity constants in [4] include the following

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theorem which uses the idea of separation of a set. If $T \subseteq X$, then the separation of T is defined by

$$\operatorname{sep} T = \inf\{\rho(x, y): x, y \in T, x \neq y\}.$$

THEOREM 1 (Dunham). *Let f_k, $k = 1, ...$, be a sequence of functions in $C[a, b]$ such that the set of extreme points, E_k, of $f_k - B(f_k)$, $k = 1, ...$, consists of precisely $n+1$ points for each $k = 1, ...$, and $\lim_{k \to \infty} \operatorname{sep} E_k = 0$. Then $\lim_{k \to \infty} \gamma(f_k) = 0$, i.e., the set $\{f_k: k = 1, ...\}$ does not have a uniform strong unicity constant.*

All of the above results give just necessary or just sufficient conditions for the existence of a uniform strong unicity constant. Also, in Theorem 1, the conclusion need not follow without the assumption that each E_k is of minimal cardinality. For example on $[0, 1]$ with $M = \pi$, polynomials of degree one or less, and for each $k = 1, 2, ...$, if $f_k(x)$ is the piecewise linear function defined by

$$f_k(x) = \begin{cases} -1 & \text{if } x = 1/3 - 1/k, 2/3 - 1/k \\ 0 & \text{if } x = 0, 1 \\ +1 & \text{if } x = 1/3 + 1/k, 2/3 + 1/k \end{cases} \tag{1.1}$$

then the set of functions $\{f_k\}$ has a uniform strong unicity constant even though $\lim_{k \to \infty} \operatorname{sep} E_k = 0$. If in the same setting f_k is the piecewise linear function defined by

$$f_k(x) = \begin{cases} -1 & \text{if } x = 3/4 - 1/k \\ 0 & \text{if } x = 0, 3/8, 1 \\ +1 & \text{if } x = 1/4, 1/2, 3/4 + 1/k \end{cases} \tag{1.2}$$

then $\lim_{k \to \infty} \operatorname{sep} E_k = 0$ and the set $\{f_k\}$ does not have a uniform strong unicity constant. Both examples can be verified by using the characterization of strong unicity constants in (2.1).

The results in this paper completely determine (see Theorem 8) whether a given set $S \subseteq C[a, b]$ has a uniform strong unicity constant by using the notion of limit extremals. Moreover only Theorems 7 and 8 assume X is an interval. The paper's results contain all the above mentioned previous results on the problem of uniform strong unicity constants. In Section 4 the paper's results are used to show that the class of rational functions studied by Rivlin does not have a uniform strong unicity constant.

2. Preliminaries

For $f \in C(X)$, X compact metric, and M a Haar subspace of dimension $n > 1$, a critical point set is a set of $n+1$ points $x_1, ..., x_{n+1}$ such that there exist signs $\sigma_1, ..., \sigma_{n+1}$ and numbers $\theta_1, ..., \theta_{n+1}$, $\theta_i > 0$, such that $(f - B(f))(x_i) = \sigma_i \| f - B(f) \|$, $i = 1, ..., n+1$, and for each $j = 1, ..., n$,

$$0 = \sum_{i=1}^{n+1} \theta_i \sigma_i m_j(x_i).$$

Let F_δ denote the set of functions $f \in C(X)$ such that each f has a critical point set with separation $\geqslant \delta$. Then Dunham [4] proved the following results for $X = [a, b]$, and the result, with essentialy the same proof, holds for any compact metric space X.

THEOREM 2 (Dunham). *Let M be a Haar set of dimension n in $C(X)$, X compact metric. Then F_δ has a uniform strong unicity constant.*

The following characterization of $\gamma(f)$ from [2] will be used:

$$\gamma(f) = \inf_{\substack{m \in M \\ \|m\| = 1}} \max_{x \in E(f)} \frac{f(x) - B(f)(x)}{\| f - B(f) \|} m(x). \tag{2.1}$$

Let $E(f)$ denote the set of extreme points of $f - B(f)$,

$$E(f) = \{x \in X : | f(x) - B(f)(x) | = \| f - B(f) \| \}$$

and let $E^+(f)(E^-(f))$ denote the positive (resp. negative) extreme points where $(f - B(f))(x)$ has value $\| f - B(f) \|$ (resp. $-\| f - B(f) \|$). Let $|E|$ denote the cardinality of the set E and if $S \subseteq C(X)$, the set of extreme point sets $E(S)$ is defined by

$$E(S) = \{E(f) : f \in S\}.$$

DEFINITION. Let $S = \{f_k\}$ be a sequence of functions in $C(X)$. A point $x \in X$ is called a *$+$limit extremal* of S if for each k there exists $x_k^+ \in E^+(f_k)$ such that $\lim_{k \to \infty} x_k^+ = x$. A *$-$limit extremal* is defined similarly. A point $x \in X$ is a *$\pm$limit extremal* of S if for each k there exist $x_k^+ \in E^+(f_k)$ and $x_k^- \in E^-(f_k)$ such that $\lim_{k \to \infty} x_k^+ = \lim_{k \to \infty} x_k^- = x$.

In example (1.1) the point $x = 3/4$ was a $\pm$limit extremal.

All reference to the convergence of subsets of X refers to convergence of sets in the compact metric space consisting of the nonempty, closed subsets of X with the Hausdorff metric. For subsets $A, B \subseteq X$ the Hausdorff metric $d(A, B)$ is defined by

$$d(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \rho(a, b), \sup_{b \in B} \inf_{a \in A} \rho(a, b)\}.$$

When a sequence $\{E(f_k)\}$ of extreme point sets converges to a set E^0 it follows that E^0 is a maximal cluster point of the sequence.

DEFINITION. Let $S = \{f_k\}$ be a sequence of functions in $C(X)$ such that $\{E(f_k)\} \to E^0$. Then E^0 is said to contain a *limit critical point set* if it contains $n+1$ distinct limit extremals $x_1, ..., x_{n+1}$ and for each k there is a critical point set $\{x_1(k), ..., x_{n+1}(k)\}$ for f_k such that

$$\lim_{k \to \infty} x_i(k) = x_i, \qquad i = 1, ..., n+1.$$

If $X = [a, b]$ then the critical point sets are alternation sets and limit critical point sets will be called *limit alternation sets.*

3. RESULTS IN $C(X)$

The hypothesis of Theorem 1 that $\lim_{k \to \infty} \operatorname{sep} E_k = 0$ implies that there is a limit critical point set of cardinality less than or equal to n. A small cardinality for E^0 by itself is enough to guarantee the nonexistence of uniform strong unicity.

THEOREM 3. *If $S = \{f_k\}$ is a sequence in $C(X) \backslash M$, $\{E(f_k)\} \to E^0$, and $|E^0| \leqslant n-1$, then S does not have a uniform strong unicity constant.*

Proof. By interpolation there exists a function $p \in M$ with $\|p\| = 1$ and $p = 0$ on E^0. Given $\varepsilon > 0$ let N be a neighborhood of E^0 such that $|p(x)| < \varepsilon$ if $x \in N$. For k sufficiently large, $E(f_k) \subseteq N$ and by (2.1)

$$\gamma(f_k) \leqslant \max_{x \in E(f_k)} \operatorname{sgn}(f_k - B(f_k))(x)\, p(x) \leqslant \sup_{x \in N} |p(x)| \leqslant \varepsilon.$$

Since this was for any $\varepsilon > 0$, the proof is complete.

THEOREM 4. *If $S = \{f_k\}$ is a sequence in $C(X) \backslash M$, $\{E(f_k)\} \to E^0$, $|E^0| = n$, and not every point of E^0 is a $\pm$limit extremal of S, then S does not have a uniform strong unicity constant.*

Proof. Fix a point $x \in E^0$ where x is not a $\pm$limit extremal of S. Let U be a neighborhood of x and $\{f_k\}$ be a subsequence (renamed $\{f_k\}$) such that $U \cap E^-(f_k) = \varnothing$ for all k (the case $U \cap E^+(f_k) = \varnothing$ is similar). Let $p \in M$ satisfy $\|p\| = 1$, $p = 0$ on $E^0 \setminus \{x\}$, and $p(x) < 0$. Reduce U so that $p < 0$ on U. For any $\varepsilon > 0$, let N be a neighborhood of $E^0 \setminus \{x\}$ on which $|p| \leqslant \varepsilon$. Applying (2.1) we are done.

THEOREM 5. *If $S = \{f_k\}$ is a sequence in $C(X) \setminus M$, $\{E(f_k)\} \to E^0$, and E^0 contains n $\pm$limit extremals of S, then S has a uniform strong unicity constant.*

Proof. Suppose to the contrary that $\inf \gamma(f_k) = 0$ and let $\{f_k\}$ be a subsequence (renamed $\{f_k\}$) such that $\lim_{k \to \infty} \gamma(f_k) = 0$. We can assume without loss of generality that $B(f_k) = 0$ and $\|f_k\| = 1$ for each k. We still have $\{E(f_k)\} \to E^0$. Since by (1.1)

$$\lim_{k \to \infty} \gamma(f_k) = \lim_{k \to \infty} \inf_{\|m\| = 1} \max_{x \in E(f_k)} f_k(x)\, m(x) = 0,$$

for any k there exists a function $m_k \in M$ such that (relabeling if necessary and using a subsequence of $\{f_k\}$ if necessary)

$$\max_{x \in E(f_k)} f_k(x)\, m_k(x) \leqslant 1/k$$

with $\|m_k\| = 1$. Fix x_j, a $\pm$limit extremal in E^0, and let $x_{jk}^+ \in E^+(f_k)$ and $x_{jk}^- \in E^-(f_k)$ satisfy

$$\lim_{k \to \infty} x_{jk}^+ = x_j = \lim_{k \to \infty} x_{jk}^- .$$

Then $m_k(x_{jk}^+) \leqslant 1/k$ and $-1/k \leqslant m_k(x_{jk}^-)$. Since $\{m_k\}$ is a uniformly bounded sequence in M, there exists $\bar{m} \in M$ such that $\{m_k\}$ (using a subsequence and relabeling if necessary) converges to $\bar{m}$ with $\|\bar{m}\| = 1$. Since the set $\{m_k\}$ is uniformly equicontinuous,

$$\lim_{k \to \infty} m_k(x_{jk}^+) = \bar{m}(x_j) = \lim_{k \to \infty} m_k(x_{jk}^-).$$

Thus $\bar{m}(x_j) = 0$. Hence $\bar{m}$ has at least n zeros since there are at least n $\pm$limit extremals. Thus $\bar{m} = 0$ which contradicts $\|\bar{m}\| = 1$ and the proof is complete.

Remark. Theorem 5 shows that the example in (1.1) has a uniform strong unicity constant since $n = 2$ and $\frac{1}{3}$ and $\frac{2}{3}$ are $\pm$limit extremals.

THEOREM 6. *If $S = \{f_k\}$ is a sequence in $C(X) \setminus M$, $\{E(f_k)\} \to E^0$, and E^0 contains a limit critical point set, then S has a uniform strong unicity constant.*

Proof. Suppose to the contrary that $\inf_k \gamma(f_k)=0$. Let $\{f_k\}$ be a subsequence (renamed $\{f_k\}$) such that $\lim_{k\to\infty}\gamma(f_k)=0$ and assume without loss of generality that $\|f_k\|=1$ and $B(f_k)=0$ for each $k=1, \ldots$.

Let $\{x_1, \ldots, x_{n+1}\}$ be a limit critical point set in E^0 with separation $\eta>0$ and let $\{x_1^{(k)}, \ldots, x_{n+1}^{(k)}\}=A(f_k)$ be a critical point set for f_k for each k, where $\lim_{k\to\infty} x_i^{(k)}=x_i$, $i=1, \ldots, n+1$.

Then for k large enough $\operatorname{sep} A(f_k)\geqslant \eta/2>0$ and thus by Theorem 2 we are led to a contradiction and the proof is complete.

3. Results in $C[a, b]$

For the remainder of the paper $X=[a, b]$.

THEOREM 7. *If $S=\{f_k\}$ is a sequence in $C[a,b]\backslash M$, $\{E(f_k)\}\to E^0$, $|E^0|\geqslant n+1$, and E^0 does not contain a limit alternation set for any subsequence of S, then S does not have a uniform strong unicity constant.*

Proof. By extraction of subsequences and relabeling, we may assume that E^0 contains r $\pm$limit extremals of $\{f_k\}$, $y_1<\cdots<y_r$, and no other point of E^0 is a $\pm$limit extremal of a subsequence of $\{f_k\}$. By $|E^0|\geqslant n+1$, $r\leqslant n-1$. Let $\varepsilon>0$. By the uniform equicontinuity of the unit ball of M, there exists $\delta>0$ such that $p\in M$, $\|p\|=1$, and $|x-y|\leqslant\delta$ implies $|p(x)-p(y)|\leqslant\varepsilon$. We shall select a sign $\sigma=\pm 1$, a subsequence relabeled $\{f_k\}$, and s points $z_1<\cdots<z_s$, in $[a, b]$ with $s\leqslant n-1$ satisfying

(i)	$x\in[a, z_1-\delta]\cap E(f_k)$,	$\sigma f_k(x)>0$
(ii)	$x\in[z_i+\delta, z_{i+1}-\delta]\cap E(f_k)$,	$(-1)^i\,\sigma f_k(x)\ (i=1, \ldots, s-1)$
(iii)	$x\in[z_s+\delta, b]\cap E(f_k)$,	$(-1)^s\,\sigma f_k(x)>0.$

Once we have accomplished this, Theorem 5.2 in [7] yields $p\in M$ with $\|p\|=1$ where $\sigma p\leqslant 0$ on $[a, z_1]$, $(-1)^i\sigma p\leqslant 0$ on $[z_i, z_{i+1}]$ $(i=1, \ldots, s-1)$, and $(-1)^s\sigma p\leqslant 0$ on $[z_s, b]$. By (2.1) we would then have $\gamma(f_k)\leqslant\varepsilon$ for all k.

Choose the first interval $[a, y_1)$, (y_1, y_2), ..., $(y_r, b]$ that contains a point of E^0. Since $r\leqslant n-1$, one indeed exists. Suppose that (y_j, y_{j+1}) is the first such interval. (There is virtually no difference in the consideration when $[a, y_1)$ or $(y_r, b]$ is the first such interval). Let $z_1=y_1, \ldots, z_j=y_j$. Choose a subsequence and relabel so that $E(f_k)\cap[a, y_j]\subseteq\bigcup_{i=1}^{j}(y_i-\delta, y_i+\delta)$ for all k. If $(y_j, y_j+\delta)\cap E^0\neq\varnothing$ choose x in this set. Otherwise, let x be the smallest element of $(y_j, b]\cap E^0$. Either way, choose a subsequence of $\{f_k\}$ so that (for instance) x is s $+$limit extremal of $\{f_k\}$. Observe that x is not a $-$limit extremal of any subsequence of $\{f_k\}$. Now let z_{j+1} be the smallest element of $[x, b]$ that is a $-$limit extremal of a subsequence of $\{f_k\}$. If no such z_{j+1} exists, then we can choose a subsequence and relabel so that $f_k > 0$ on $[x, b] \cap E(f_k)$ for all k and the construction would be complete. If z_{j+1} does exist, choose a subsequence and relabel so that z_{j+1} is a $-$limit extremal of $\{f_k\}$. We may further choose a subsequence and relabel so that $f_k > 0$ on $[z_j + \delta, z_{j+1} - \delta] \cap E(f_k)$ for all k. Now choose z_{j+2} to be the smallest element of $[z_{j+1}, b]$ which is a $+$limit extremal of a subsequence of $\{f_k\}$. If none exists, we would be done as above. Otherwise, perform the same extractions as above. We continue in this fashion alternating signs. The process must terminate with $s \leqslant n-1$; for otherwise, $z_1 < \cdots < z_j < x < z_{j+1} < \cdots < z_n$ would constitute a limit alternation set for a subsequence of the original S.

We summarize the previous results now in Theorem 8 which completely characterizes the sets $S \subseteq C[a, b]$ which have uniform strong unicity constants. It should be observed that since $\gamma(m) = 1$ for each $m \in M$, a set $S \subseteq C(X)$ fails to have a uniform strong unicity constant precisely when $S \setminus M$ does. Also for any $m \in M$, $E(m) = X$ and thus the sets E^0 of the next theorem must arise from functions not in M. Thus the next theorem could be stated for $S \subseteq C[a, b]$ rather than for $S \subseteq C[a, b] \setminus M$.

THEOREM 8. *A set $S \subseteq C[a, b] \setminus M$ does not have a uniform strong unicity constant if and only if S contains a sequence $\{f_k\}$ with $\{E(f_k)\} \to E^0$ where one of the following holds:*

(i) $|E^0| \leqslant n-1$,

(ii) $|E^0| = n$ *and E^0 contains a point which is not a $\pm$limit extremal of $\{f_k\}$,*

(iii) $|E^0| \geqslant n+1$ *and E^0 does not contain a limit alternation set for any subsequence of $\{f_k\}$.*

Proof. Theorems 3, 4, and 7 show that any one of the above conditions gives a nonuniform strong unicity constant. If S does not have a uniform strong unicity constant, i.e., $\inf_{f \in S} \gamma(f) = 0$, then there exists a sequence $\{f_k\}$ in S such that $\lim_{k \to \infty} \gamma(f_k) = 0$. Then there will be a subsequence (renamed $\{E(f_k)\}$) of $\{E(f_k)\}$ which converges to a set E^0. If none of the above three conditions held then Theorem 5 and 6 would ensure that $\{f_k\}$ had a uniform strong unicity constant.

Remark. The result of Henry and Schmidt [5] and Paur and Roulier [10] follows from Theorem 6 for if there is some sequence $\{f_k\}$, $f_k \in S \subseteq C[a, b]$, $S \cap M = \varnothing$, and S compact, then they showed that any cluster point of $E(f_k)$ contains an alternation set. Cline's result [3] for all of $C[a, b]$ follows from Theorem 3 by considering a sequence of functions

$\{f_k\}$, $f_k \in C[0, 1]$, such that all the extreme points $E(f_k) \subseteq [1/2 - 1/k, 1/2 + 1/k]$ and thus the only cluster point of $E(f_k)$ would be $E^0 = \{1/2\}$. Bartelt's result [1] for X finite follows immediately from Theorem 2.

4. A Class of Rational Functions

In [11], T. J. Rivlin studied a set of rational functions

$$S = \{f(t, x): 0 < t < 1\} \subseteq C[-1, 1],$$

where a and b are integers, $a > 0$, $b \geqslant 0$, $n_k = ak + b$, $k = 1, ...,$ and T_k is the kth degree Chebyshev polynomial

$$f(t, x) = \sum_{k=0}^{\infty} t^k T_{n_k}(x).$$

By applying Theorem 4 in the special case $b = 0$ and Theorem 7 in case $b \neq 0$ we prove:

THEOREM 9. *Let S be the set of rational functions above, and approximate from π_n the polynomials of degree $\leqslant n$, for any $n \geqslant a + b$ with $n > 1$. Then S does not have a uniform strong unicity constant.*

For the proof we need the results from [11],

$$f(t, x) = \frac{T_b(x) - tT_{|a-b|}(x)}{1 + t^2 - 2tT_a(x)};$$

for $j = n_k, n_k + 1, ..., n_{K+1} - 1$ the best jth degree polynomial approximate for f on $[-1, 1]$ is

$$B_{ak+b}(x) = \sum_{l=0}^{k} t^l T_{al+b}(x) + \frac{t^{k+2}}{1 - t^2} T_{ak+b}(x);$$

the error function

$$\begin{aligned} e_j f(x) &= f(t, x) - B_{ak+b}(x) \\ &= \frac{t^{k+1}}{1 - t^2} \frac{A(\theta)}{B(\theta)}, \end{aligned}$$

where $A(\theta)/B(\theta) = \cos n_k(\theta + \phi)$ and where $x = \cos \theta$,

$$\cos\phi = \frac{-2t + (1+t^2)\cos(a\theta)}{1 + t^2 - 2t\cos(a\theta)}$$

$$\sin\phi = \frac{(1-t^2)\sin(a\theta)}{1+t^2-2t\cos(a\theta)},$$

and $A(\theta)/B(\theta) = \pm 1$ alternately at $n_{k+1}+1$ points.

From [6] we know that these $n_{k+1}+1$ points are $x_0 = 1$, $x_{n_{k+1}} \equiv 1$, and the $n_{k+1}-1$ roots of

$$g(t, x) = aT'_{n_k}(x)[-2t + (1+t^2)\, T_a(x)] + n_k T_{n_k}(x)(1-t^2)\, T'_a(x)$$

and we know

$$g_x(t, x_i) = \frac{(-1)^{+i}\, an_k[n_k(1+t^2-2tT_a(x_i)) + a(1-t^2)]}{x_i^2 - 1}.$$

Now it is easy to check that $\operatorname{sgn} e_j(f)(1) = 1$,

$$\operatorname{sgn} e_j(f)(x_i) = (-1)^{+i}, \qquad i = 0, ..., n_{k+1},$$

and

$$g_t(x_i, t) = \frac{-2aT'_{n_k}(x_i)[1+t^2-2tT_a(x_i)]}{1-t^2}$$

and thus considering x_i as a function of t, $0 < t < 1$,

$$\frac{dx_i}{dt} = \frac{2aT'_{n_k}(x_i)[1+t^2-2tT_a(x_i)][x_i^2-1]}{(1-t^2)(-1)^{+i}\, an_k[n_k(1+t^2-2tT_a(x_i)) + a(1-t^2)]}. \tag{4.1}$$

Also $g(0, x) = aT'_{n_k}(x)\, T_a(x) + n_k T_{n_k}(x)\, T'_a(x) = an_k/n_{k+1}\, T'_{n_{k+1}}(x)$ and $g(1, x) = 2aT'_{n_k}(x)[T_a(x) - 1]$.

Since the roots of $g(x, t)$ are continuous functions of t, we have $x_i(0) = z_i$ where $T'_{n_{k+1}}(z_i) = 0$ while $x_i(1)$ is a root of $g(1, x)$.

Since $T_a(x) - 1$ has $[a/2]+1$ roots (always including 1 and including -1 if a is even) $g(1, x)$ has at most $n_k + [a/2]$ distinct roots in $[-1, 1]$. So as t varies from 0 to 1, the $n_{k+1}+1$ extreme points of $e_j(f)$ coalesce into at most $n_k + [a/2]$ points.

Proof of Theorem. Assume first that $b \neq 0$ and that $T_a(x) - 1$, $T'_{n_k}(x)$, and $T'_{n_{k+1}}(x)$ have no roots in common. Let

$$-1 < z(n_{K+1} - 1) < \cdots < z_1 < 1$$

be the roots of $T'_{n_{k+1}}(x)$ where

$$z(i)=\cos(i\pi/n_{k+1}), \qquad i=1, ..., n_{k+1}-1$$

and let

$$w(n_k-1)< \cdots <w(1)$$

be the roots of T'_{n_k} where

$$w(i)=\cos(i\pi/n_k), \qquad i=1, ..., n_k-1,$$

and let

$$q\left(\left[\frac{a}{2}\right]\right)< \cdots <a(1)<q(0)=1$$

be the roots of $T_a(x)-1$ where

$$q(i)=\cos(2i\pi/a), \qquad i=0, ..., \left[\frac{a}{2}\right]$$

and let

$$-1\leqslant\lambda\left(\left[\frac{a+1}{2}\right]\right)< \cdots <\lambda(1)<1, \qquad \left(\lambda\left(\left[\frac{a+1}{2}\right]\right)=-1 \text{ if } a \text{ is odd}\right)$$

be the roots of $T_a(x)+1$ where

$$\lambda(j)=\cos((2j-1)\pi/a, \qquad j=1, ..., \left[\frac{a+1}{2}\right].$$

Then from [6] in this setting we know

$$M_{n_k}f(t, x)\leqslant M_{n_k+1}\leqslant \cdots \leqslant M_{n_k+1}-1,$$

where $M_n=1/\gamma_n$ and γ_n is the strong unicity constant when approximating from Π_n. Thus it suffices to show

$$\sup_{0<t<1} M_{n_k}(f(t, x))=\infty.$$

Let $-1<u(a-1)< \cdots <u(1)<1$ be the interior extreme points of $T_a(x)$. So $u(1)$, $u(3)$, ... etc., are the $\lambda(i)$ and $u(2)$, $u(4)$, ... are the $q(i)$ ($u(i)=\cos(i\pi/a)$, $i=1, ..., a-1$). Let I_1 be the largest integer such that $I_1/n_{k+1}<1/a$, I_2 the largest integer such that $(I_1+I_2)/n_{k+1}<2/a$, ..., and I_{a-1} the largest integer such that $\sum_{i=1}^{a-1} I_i/n_{k+1} < (a-1)/a$. This leads to the following ordering of the zeros under consideration:

$$\begin{aligned}
&1 > z_1 > w_1 > \cdots > w(I_1-1) > z(I_1) > u(1) > z(I_1+1) > w(I_1) \\
&> \cdots > w(I_1+I_2-2) > z(I_1+I_2) > u(2) > z(I_1+I_2+1) > w(I_1+I_2-1) \\
&> \cdots > w(I_1+\cdots+I_i-i) > z(I_1+\cdots+I_i) > u(i) > z(I_1+\cdots+I_i+1) \\
&> w(I_1+\cdots+I_i-i+1) > \cdots > w(I_1+\cdots+I_{a-1}-(a-1)) \\
&> z(I_1+\cdots+I_{a-1}) \\
&> u(a-1) > A(I_1+\cdots+I_{a-1}+1) \\
&> w(I_1+\cdots+I_{a-1}-(a-1)+1) > \cdots \\
&> z(n_{k+1}-2) > w(n_k-1) > z(n_{k+1}-1) > -1.
\end{aligned}$$

To verify the ordering observe that by the definition of I_1 we have

$$I_1/n_{k+1} < 1/a < (I_1+1)/n_k + 1.$$

Thus

$$I_1 < n_{k+1}/a < I_1+1 \qquad \text{and} \qquad I_1 < K+1+b/a < I_1+1.$$

Thus

$$I_1 a - ak - a - b < 0,$$

hence

$$I_1 ak + I_1 a + I_1 b - ak - a - b < I_1 ak + I_1 b,$$

hence

$$I_1 n_{k+1} - n_{k+1} < I_1 n_k,$$

hence

$$(I-1)/n_k < I_1/n_{k+1},$$

hence

$$w(I_1-1) > z(I_1).$$

On the other hand

$$I_1 > k + b/a,$$

hence

$$I_1 a > ak + b,$$

hence

$$I_1 ak > I_1 a + I_1 b > I_1 ak + I_1 b + ak + b,$$

hence

$$I_1/n_k > (I_1 + 1)/n_{k+1},$$

hence

$$z(I_1 + 1) > w(I_1).$$

The verification of the rest of the ordering can be done in a similar way using induction.

Let $x(1), ..., x(n_{k+1} - 1)$ be the interior extreme points of $e_j(f)(x)$

$$-1 < x(n_{k+1} - 1) < \cdots < x(1) < 1.$$

Then the x_i fit into the previous ordering as follows,

$$1 > x(1) > z(1), \qquad w(1) > x(2) > z(2),$$
$$x(I_1 - 1) > z(I_1 - 1) > w(I_1 - 1) > x(I_1) > z(I_1),$$

and

$$\begin{aligned} z(I_1 + \cdots + I_{2j}) > x(I_1 + \cdots + I_{2j}) > u(2j) > x(I_1 + \cdots + I_{2j+1}) \\ > z(I_1 + \cdots + I_{2j+1}) > w(I_1 + \cdots + I_{2j} - 2j + 1) \\ > x(I_1 + \cdots + I_{2j+1}) > \cdots \end{aligned} \tag{4.2}$$

This follows easily from (4.1). Furthermore as $t \to 1$

$$x(1) \to 1, \qquad x(2) \to w(1), ..., \qquad x(I_1) \to w(I_1 - 1)$$
$$x(I_1 + 1) \to w(I_1), ..., \qquad x(I_1 + I_2 - 1) \to w(I_1 + I_2 - 2), \qquad x(I_1 + I_2) \to u(2)$$
$$x(I_1 + I_2 + 1) \to u(2), \qquad \text{etc.}$$

Thus note that no $w(i)$ is a $\pm$limit extremal.

Let $A(t)$ be an alternant for $f(t, x)$. Suppose for some j that $A(t)$ contains $x(I_1 + \cdots + I_{2j})$ and $x(I_1 + \cdots + I_{2j+1})$. Then as $t \to 1$, both $x(I_1 + \cdots + I_{2j})$ and $x(I_1 + \cdots + I_{2j} + 1)$ tend to $u(2j)$. Thus $A(1)$ has cardinality at most $n_k + 1$ and thus it is not a limit alternation set.

Now suppose that the above does not happen for any j. Then we have the following three possibilities:

(i) $A(t)$ contains $x(I_1 + \cdots + I_{2j})$ but not $x(I_1 + \cdots + I_{2j} + 1)$. In this case, to preserve alternation, $A(t)$ cannot contain $x(I_1 + \cdots + I_{2j} + 2)$.

(ii) $A(t)$ contains $x(I_1 + \cdots + I_{2j+1})$ but not $x(I_1 + \cdots + I_{2j})$. Then $A(t)$ does not contain $x(I_1 + \cdots + I_{2j} - 1)$.

(iii) $A(t)$ contains neither $x(I_1 + \cdots + I_{2j})$ nor $x(I_1 + \cdots + I_{2j} + 1)$.

In any case, for each $u(2j)$, $A(t)$ does not contain two of the $x(j)$. Since there are $[a/2] - 1$ of the $u(2j)$'s if a is even ($[a/2]$ if a is odd), there are $a - 2$ of the interior $x(i)$ that are omitted from $A(t)$ if a is even ($a - 1$ if a is odd). Thus $A(t)$ contains only $n_{k+1} - 1 - (a - 2) = n_k + 1$ interior points if a is even (n_k if a is odd). Furthermore $A(t)$ must include $x(1)$ and $x(n_{k+1} - 1)$. Thus for $A(t)$ to be an alternant if a is even, $A(t)$ must include either 1 or -1. But as $t \to 1$, $x_1 \to 1$ and $x(n_{k+1} - 1) \to -1$. So $A(1)$ has cardinality at most $n_k + 1$. If a is odd, $A(t)$ must include both 1 and -1 and again $A(1)$ has cardinality at most $n_k + 1$.

In either case $A(1)$ is not a limit alternation set and consequently E^0 does not contain a limit alternation set and the result follows from Theorem 7.

Now if $b \neq 0$ and $T_a(x) - 1$, $T'_{n_k}(x)$, and $T'_{n_{k+1}}(x)$ do have some roots in common, the argument is similar to the preceding case and uses the fact that if x is a common root of T'_{n_k} and $T_a(x) - 1$, then x is also a root of $T'_{n_{k+1}}$ and if Z is the root of $T'_{n_{k+1}}$ closest to x then $z \to x$ as $t \to 1$. Also in (4.2) some of the strict inequalities $>$ become $\geqslant$.

Finally if $b = 0$, then all the interior roots of $T_a(x) - 1$ are roots of $T'_{n_k}(x) = T'_{ak}(x)$. Thus $g(1, x)$ has only $n_k - 1$ interior roots and E^0 has cardinality $n_k + 1$. Since no root f $T'_{n_k}(x)$ that is not a root of $T_a(x) - 1$ can be a $\pm$limit extremal the result follows from Theorem 4.

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